

# Quantum statistical mechanics of ideal gas obeying fractional exclusion statistics: a systematic study\*

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**Abstract.** The quantum statistical mechanics of an ideal gas with a general free-particle energy obeying fractional exclusion statistics are systematically investigated in arbitrary dimensions. The pressure relations, the relation between pressure and internal energy, the equation of state, as well as the thermodynamic properties are thoroughly discussed. Some novel results are obtained.

**PACS.** 05.30.-d Quantum statistical mechanics – 71.10.-w Theories and models of many electron systems – 71.15.-m Methods of electronic structure calculations

## 1 Introduction

In standard textbooks, quantum statistics usually refers to the well-known Bose and Fermi ones. The former admits that a state under permutation is symmetric, leading the maximum occupation number of particles in one state to infinity, while the latter requires that any state under permutation should be antisymmetric, giving rise to the maximum occupation number in one state being one. These two statistics are of fundamental significance for our understanding of the real physical world nowadays. Indeed, a great number of physically basic phenomena have been successfully explained within the framework of these two celebrated statistics. Are they, however, adequate for us to describe all macroscopic properties of the nature? The answer could be no. As a matter of fact, after invention of these two statistics some people have been considering how to generalize the quantum statistics to cover many other cases. A few such generalizations have therefore been done up to date.

One generalization is the so-called parastatistics (including para-Bose and para-Fermi statistics) [1]. Such a kind of statistics is obtained in the following way: The expectation value of any observable in the permuted state is required to be the same as in the unpermuted state. It thus brings about two cases. The para-Fermi statistics allows up to a finite number (greater than one) of particles occupying one state, while the para-Bose statistics allows that the wavefunction in one state vector can be antisymmetric with respect to a finite number (greater than one) of particles but keeping the maximum occupation number to infinity. This generalization has some implications in

quantum field theories, and is extensively studied in past decades.

Another generalization is named as the intermediate statistics [2]. This statistics is obtained by simply postulating that the maximum occupation number (denoted by  $\nu$ ) of particles in a single-particle state is finite, which naturally interpolates between Fermi ( $\nu = 1$ ) and Bose ( $\nu = \infty$ ) statistics. However, there is neither a mathematical basis for the symmetry properties of wavefunctions nor any generalized field quantization scheme available for intermediate statistics. One of implications of this statistics can be attributed to magnons in the Heisenberg model of magnets. As is well-known, the maximum number of magnons in a system with a given number of spins is finite, suggesting that magnons are not exactly bosons.

There are other generalizations (*e.g.*, the  $q$ -deformed Fermi and Bose statistics (see, *e.g.* Refs. [3]) motivated by the quantum group theory, *etc.*). The most recent and more physical generalization is the famous fractional exclusion statistics (FES), first proposed by Haldane [4] and then realized by Wu [5]. Inspired by studies in the fractional quantum Hall systems and in one-dimensional (1d) exactly solvable models, and by considering many particle systems with finite dimensional Hilbert spaces where the dimensionality of the single-particle Hilbert space depends *linearly* on the total number of particles, Haldane [4] defined the exclusion statistics through the changes of the dimension of the single-particle space and the number of particles. By interpolating the statistical weights of Bose and Fermi statistics, Wu [5] in his seminar paper derived the distribution function of FES, which is usually called as Haldane-Wu distribution function in literature. Recently it becomes aware that the physical realizations of FES and associated generalized ideal gas can be reached in 1d integrable models with long-range interactions of quantum fluids where the particles are shown to obey FES, in the

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lowest Landau level where anyons are shown to obey FES, and in the fractional quantum Hall effect where the quasi-holes and the quasi-electrons obey FES [6]. Consequently, since the appearance of Wu's remarkable paper a number of works on FES have been done [7–15]. However, some basic issues, for instance, the fundamental pressure inequalities which are well known for Bose and Fermi statistics, a general relation between the pressure and internal energy (Bernoulli equation), as well as the thermodynamic properties, *etc.* on quantum statistical mechanics (QSM) of FES still need to be unambiguously addressed. As FES has essential implications in low-dimensional physical systems, a thorough discussion on it is really interesting and necessary.

The outline of this paper is as follows. In Section 2 the definition and the basic formulae of FES are briefly recalled. The pressure relations, a relation between the pressure and internal energy, the equation of state, as well as the thermodynamic properties are presented in Sections 3–6, respectively. Finally, a summary of results is given.

## 2 Fractional exclusion statistics

Let us first briefly recall the definition and some basic formulae of FES in this section, which will be used in the subsequent discussions. By adopting a state-counting definition, Wu [5] managed to write down the statistical weight, say, the number of quantum states of  $N$  identical particles obeying FES occupying a group of  $G$  states, as

$$W = \frac{[G + (N-1)(1-g)]!}{N![G - gN - (1-g)]!}, \quad (2.1)$$

where the statistics parameter  $g$  represents the number of states that one particle can “occupy”, or the ability of a particle to exclude other particles in “occupying single-particle state”, with  $g = 0$  corresponding to usual bosons and  $g = 1$  fermions. (Hereafter we only consider  $0 \leq g \leq 1$ ). By assuming that an ideal gas, where every single-particle state of species  $i$  has the same energy  $\epsilon_i$ , is a system with the total energy taking a simple sum

$$E = \sum_i N_i \epsilon_i, \quad (2.2)$$

and using the standard argument of QSM, he obtained the grand partition function

$$\Xi = \sum_{\{N_i\}} W(\{N_i\}) \exp\{-\beta \sum_i N_i (\epsilon_i - \mu)\}, \quad (2.3)$$

with  $\beta = 1/k_B T$  the inverse temperature ( $k_B$  the Boltzmann constant) and  $\mu$  the chemical potential. The most-probable distribution of  $n_i$ , the average “occupation number” defined by  $N_i/G_i$ , for identical particles was shown [5] to be determined by

$$n_i = \frac{1}{\omega(e^{\beta(\epsilon_i - \mu)} + g)}, \quad (2.4)$$

where the function  $\omega(\xi)$  satisfies the functional equation

$$\omega(\xi)^g [1 + \omega(\xi)]^{1-g} = \xi \equiv e^{\beta(\epsilon - \mu)}. \quad (2.5)$$

Equation (2.4) is the so-called Haldane-Wu distribution function. The thermodynamic potential  $\Omega$  and the total number of particles  $N$  can thus be obtained by

$$\Omega = -PV = -k_B T \sum_i G_i \log\left(1 + \frac{1}{\omega(\xi_i)}\right), \quad (2.6)$$

$$N = \sum_i G_i \frac{1}{\omega(\xi_i) + g}. \quad (2.7)$$

We note that the above equations are formulated in the representation of states. By recognizing the fact that the grand partition function at the most probable distribution can be factorizable, like the usual Bose and Fermi cases, the summation over states in above equations can be converted into that over momentum [15]. The average occupation number in momentum space can be written down as

$$\langle n_{\mathbf{p}} \rangle = \frac{1}{\omega_{\mathbf{p}} + g} \quad (2.8)$$

with the function  $\omega_{\mathbf{p}}$  satisfying

$$\omega_{\mathbf{p}}^g (1 + \omega_{\mathbf{p}})^{1-g} = e^{\beta(\epsilon_{\mathbf{p}} - \mu)}. \quad (2.9)$$

The thermodynamic potential and the total number of particles can then become as

$$\Omega = -PV = -k_B T \sum_{\mathbf{p}} \log\left(1 + \frac{1}{\omega_{\mathbf{p}}}\right), \quad (2.10)$$

$$N = \sum_{\mathbf{p}} \frac{1}{\omega_{\mathbf{p}} + g}. \quad (2.11)$$

These are basic equations useful in the subsequent analyses.

## 3 Pressure relations

It is well-known that for standard Bose and Fermi statistics the corresponding pressures ( $P_B$  and  $P_F$ ) of the ideal gases satisfy the following inequality [16]

$$P_B < P_{cl} < P_F \quad (3.1)$$

for the fixed density  $\rho = N/V$ , where  $P_{cl}$  is the pressure of an ideal gas obeying Maxwell-Boltzmann statistics (or classical statistics henceforth), given by

$$P_{cl} = k_B T \frac{N}{V} = \rho k_B T. \quad (3.2)$$

The physical meaning of the inequality (3.1) is as follows. The quantum effects in Bose and Fermi statistics introduce the so-called effective “statistical interactions”. The particles obeying Fermi statistics tend to expel other particles more than those obeying the classical statistics

(in the latter case no any interaction exists between particles), while the particles obeying Bose statistics tend to attract other particles, yielding the possibility of condensation of bosons. In other words, this inequality reflects the consequences of Pauli's exclusion principle and Bose-Einstein condensation.

Now let us consider if there exists a similar inequality for FES. From equation (2.10) we know that the pressure  $P$  is given by

$$P = \frac{k_B T}{V} \sum_{\mathbf{p}} \log \left( 1 + \frac{1}{\omega_{\mathbf{p}}} \right). \quad (3.3)$$

As discussed in reference [5],  $\omega_{\mathbf{p}}$  is non-negative. By utilizing the inequality  $\log(1+x) > \frac{x}{1+x}$  for  $x > 0$ , we find

$$P > \frac{k_B T}{V} \sum_{\mathbf{p}} \frac{\langle n_{\mathbf{p}} \rangle}{1 + (1-g)\langle n_{\mathbf{p}} \rangle}. \quad (3.4)$$

By equation (2.8), we have

$$\langle n_{\mathbf{p}} \rangle \leq \frac{1}{g}. \quad (3.5)$$

Incorporating inequalities (3.4) and (3.5) one can get

$$P > gP_d \quad (3.6)$$

for fixed density  $\rho$ . This inequality states that the effective interactions between particles obeying FES is more repulsive, implying an exclusion property of FES. When  $g = 1$ , Inequality (3.6) recovers the second inequality of (3.1). Therefore, the inequality (3.6) can be regarded as a generalization of (3.1).

Now we consider the free-particle energy  $\epsilon_{\mathbf{p}}$  with the following general form:

$$\epsilon_{\mathbf{p}} = c_0 \cdot \frac{\mathbf{p}^\alpha}{m} \quad (3.7)$$

with constants  $c_0$ ,  $\alpha > 0$ . When  $\alpha = 2$  and  $c_0 = 1/2$ , it reproduces the usual case of an ideal gas. When  $\alpha = 1$ , the 1d Calogero-Sutherland (CS) model [9] falls into this class with a properly scaled value of  $c_0$ . However, the following analyses hold for arbitrary  $\alpha > 0$ . The density of states (DOS) in  $d$  dimensions can thus be obtained by

$$\mathcal{D}(\epsilon) = A(\alpha, d) \epsilon^{\frac{d}{\alpha}-1} \quad (3.8)$$

$$A(\alpha, d) = \frac{S_d(m/c_0)^{d/\alpha}}{\alpha(2\pi\hbar)^d}, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}, \quad (3.9)$$

where we have transformed the summation over momentum into an integral:  $(1/V) \sum_{\mathbf{p}} \rightarrow \frac{S_d}{(2\pi\hbar)^d} \int p^{d-1} dp$ .

When  $\alpha = d$ , the DOS is constant:  $\mathcal{D}(\epsilon) = A(d, d)$ . The cases, like an ideal gas with  $\alpha = 2$  in two dimensions discussed in references [5,6], as well as the 1d CS model considered in reference [9], belong to this category. Under the condition of  $\mathcal{D}(\epsilon)$  being constant, by noticing

$(1/V) \sum_{\mathbf{p}} \rightarrow \int \mathcal{D}(\epsilon) d\epsilon$  and  $\rho = A(d, d) \int_0^\infty \frac{d\epsilon}{\omega(\epsilon) + g}$  one can get

$$\mu = k_B T \log(e^{\frac{\beta\rho}{A(d,d)}} - 1) + (g-1) \frac{\rho}{A(d,d)}. \quad (3.10)$$

Substituting (3.10) into  $\frac{\partial P}{\partial g}$  and working out the integral, we find

$$\frac{\partial P}{\partial g} = \frac{\rho^2}{2A(d,d)} > 0, \quad (3.11)$$

implying that  $P$  is a monotonically increasing function of  $g$  in this special case. In addition, equation (3.10) implies

$$\mu = g\mu_F + (1-g)\mu_B \quad (3.12)$$

with  $\mu_F = \mu(g=1)$  and  $\mu_B = \mu(g=0)$ . Considering  $\frac{\partial P}{\partial \mu} = \rho$ , it can be seen that

$$P = gP_F + (1-g)P_B \quad (3.13)$$

for fixed  $\rho$ , which is also in agreement with (3.6). Note that equation (3.13) first appeared as a comment by Suzuki [6] for  $\alpha = 2$  and  $d = 2$ . To summarize, we have the following theorem [17]:

**Theorem:** For an ideal gas obeying FES with constant density of states in  $d$  dimensions,  $\mu = g\mu_F + (1-g)\mu_B$ , and  $P = gP_F + (1-g)P_B$  for  $0 \leq g \leq 1$ .

A direct corollary of this theorem is:  $\Xi = \Xi_F^g \Xi_B^{1-g}$ , suggesting that in this special case the ideal gas with FES can be regarded as composites of usual fermions and bosons. Unlike the assertion in reference [15] where some errors remain in the proof, we should stress here that this statement could not hold true for the case where DOS is not constant.

## 4 The Bernoulli equation

Now let us discuss the general relation between the pressure and the internal energy under assumption of the free-particle energy with the form of (3.7). The internal energy  $E$  can be expressed as

$$E = \sum_{\mathbf{p}} \frac{\epsilon_{\mathbf{p}}}{\omega_{\mathbf{p}} + g} = V \int_0^\infty \frac{\mathcal{D}(\epsilon) \epsilon d\epsilon}{\omega(\epsilon) + g}. \quad (4.1)$$

By noting  $\frac{d}{d\epsilon} \log \left[ 1 + \frac{1}{\omega(\epsilon)} \right] = -\frac{\beta}{\omega(\epsilon) + g}$ , and integrating equation (4.1) by part, we obtain

$$\begin{aligned} \frac{E}{V} &= -\frac{1}{\beta} \log \left[ 1 + \frac{1}{\omega(\epsilon)} \right] \mathcal{D}(\epsilon) \epsilon \Big|_0^\infty \\ &\quad + \frac{d}{\alpha\beta} \int_0^\infty \log \left[ 1 + \frac{1}{\omega(\epsilon)} \right] \mathcal{D}(\epsilon) d\epsilon = \frac{d}{\alpha} P, \end{aligned}$$

giving rise to the Bernoulli equation

$$PV = \frac{\alpha}{d} E. \quad (4.2)$$

It can be noted that equation (4.2) is  $g$ -independent, as it should be. For  $\alpha = d$ ,  $PV = E$ , which is fit for the case of 1d CS model and the case of the usual ideal gas with  $\alpha = 2$  in two dimensions. We would like to point out here that equation (4.2) was ever derived for Bose and Fermi ideal gases by Suzuki [16] for  $d = 3$ . Now, we find that the Bernoulli equation can be extended to cover an ideal gas with FES.

In addition, using equations (3.3, 4.2, 2.9), one can find the pressure  $P(T)$  can be determined by the following functional

$$P(T) = \rho\mu(T) + C_1T + \frac{d}{\alpha}T \int^T \frac{P(x)}{x^2} dx, \quad (4.3)$$

where  $C_1$  is a temperature-independent constant that can be determined by known conditions. This equation can be solved numerically.

## 5 Equation of state

In this section, we shall discuss the equation of state, namely, the Virial expansion for an ideal gas obeying FES. Define

$$z = e^{\beta\mu} \quad (5.1)$$

as the fugacity. In parallel to Bose and Fermi statistics [18], we can expand  $\frac{P}{k_B T}$  and  $N/V$  in powers of  $z$ , and get

$$\frac{PV}{Nk_B T} = \frac{\sum_{l=1}^{\infty} b_l z^l}{\sum_{l=1}^{\infty} l b_l z^l} \quad (5.2)$$

with  $b_l$  the expansion coefficients. The Virial expansion gives

$$\frac{PV}{Nk_B T} = \sum_{l=1}^{\infty} a_l (\lambda^d \rho)^{l-1} \quad (5.3)$$

with  $\lambda$  to be determined later, and  $\lambda^d \rho < 1$ . By expanding  $\rho = \frac{N}{V}$  in the right-hand side (r.h.s.) of (5.3) in powers of  $z$ , and comparing the coefficients of equations (5.2) and (5.3), one can obtain the virial coefficients [18],

$$\begin{aligned} a_1 &= b_1 = 1, & a_2 &= -b_2, & a_3 &= 4b_2^2 - 2b_3, \\ a_4 &= -20b_2^3 + 18b_2b_3 - 3b_4, & \dots \end{aligned} \quad (5.4)$$

Now, let us calculate the coefficients  $b_l$ . By expanding the logarithmic function in  $P$  in powers of  $ze^{-\beta\epsilon_{\mathbf{p}}}$ , we have

$$\log\left(1 + \frac{1}{\omega_{\mathbf{p}}}\right) = \sum_{l=1}^{\infty} c_l (e^{-\beta\epsilon_{\mathbf{p}}} z)^l \quad (5.5)$$

for  $e^{\beta(\epsilon_{\mathbf{p}} - \mu)} > 1$ . Since this is a standard Taylor expansion, we can obtain the coefficients as

$$\begin{aligned} c_1 &= 1, & c_2 &= -\frac{1}{2!}(2g-1), \\ c_3 &= \frac{1}{3!}(3g-1)(3g-2), \\ c_4 &= -\frac{1}{4!}(4g-1)(4g-2)(4g-3), & \dots, \\ c_l &= \frac{(-1)^{l-1}}{l!} \prod_{m=2}^l [lg - (m-1)] \text{ for } l \geq 2. \end{aligned} \quad (5.6)$$

Substituting (5.5) into (3.3) yields

$$\begin{aligned} \frac{P}{k_B T} &= \int_0^{\infty} \sum_{l=1}^{\infty} c_l z^l e^{-\beta\epsilon_l} \mathcal{D}(\epsilon) d\epsilon \\ &= \sum_{l=1}^{\infty} \frac{c_l A(\alpha, d)}{(\beta l)^{\frac{d}{\alpha}}} \Gamma\left(\frac{d}{\alpha}\right) z^l = \sum_{l=1}^{\infty} \frac{b_l}{\lambda^d} z^l \end{aligned} \quad (5.7)$$

with

$$b_l = \frac{c_l}{l^{\frac{d}{\alpha}}}, \lambda^d = \frac{\beta^{\frac{d}{\alpha}}}{A(\alpha, d) \Gamma\left(\frac{d}{\alpha}\right)}. \quad (5.8)$$

Consequently, the Virial expansion coefficients are given by

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \frac{2g-1}{2^{\frac{d}{\alpha}+1}}, \\ a_3 &= \frac{(2g-1)^2}{4^{\frac{d}{\alpha}}} - \frac{(3g-1)(3g-2)}{3^{\frac{d}{\alpha}+1}}, \\ a_4 &= 20 \left( \frac{2g-1}{2^{\frac{d}{\alpha}+1}} \right)^3 - \frac{3(2g-1)(3g-1)(3g-2)}{2 \cdot 6^{\frac{d}{\alpha}}} \\ &\quad + \frac{(4g-1)(4g-2)(4g-3)}{2 \cdot 4^{\frac{d}{\alpha}+1}}, \dots, \end{aligned} \quad (5.9)$$

using (5.4). We note that the similar results are obtained for  $\alpha = 2$  [19], but our results are more general and include the results obtained in reference [19] as a special case. The equation of state can then have the form

$$\frac{PV}{Nk_B T} = 1 + a_2 \tilde{\rho} + a_3 \tilde{\rho}^2 + a_4 \tilde{\rho}^3 + \dots \quad (5.10)$$

with  $\tilde{\rho} = \lambda^d \rho$ . In the limit of  $\tilde{\rho} \rightarrow 0$ , we find that the effective ‘‘statistical interaction’’ is repulsive for  $g \geq \frac{1}{2}$  and attractive for  $g < \frac{1}{2}$ . We notice that there is a similar discussion on this point in reference [19], where he finds that no effective statistical interaction exists for semions (with  $g = \frac{1}{2}$ ). This is incorrect, because for  $g = \frac{1}{2}$ ,  $a_2 = 0$ , but  $a_3 > 0$ , suggesting the effective statistical interactions between semions are repulsive. For  $\tilde{\rho} (< 1)$  not so small, to discuss the property of effective interactions between particles obeying FES we have to take the  $a_3$  term in above equation into account, which gives if

$$g \geq \frac{-\frac{1}{4^{\frac{d}{\alpha}-1}} + \frac{1}{3^{\frac{d}{\alpha}-1}} + \frac{1}{2^{\frac{d}{\alpha}} \tilde{\rho}} - \sqrt{\left(\frac{1}{2^{\frac{d}{\alpha}} \tilde{\rho}}\right)^2 + \left(\frac{1}{3^{\frac{d}{\alpha}}}\right)^2 - \frac{1}{3^{\frac{d}{\alpha}+1} \cdot 4^{\frac{d}{\alpha}-1}}}}{2 \left(\frac{1}{3^{\frac{d}{\alpha}-1}} - \frac{1}{4^{\frac{d}{\alpha}-1}}\right)}, \quad (5.11)$$

the effective statistical interaction is repulsive. The r.h.s. of (5.11) is smaller than  $1/2$ , while  $\tilde{\rho} \rightarrow 0$ , it gives  $1/2$ . This fact implies that for the density  $\rho$  fixed there must be a critical value  $g_c$  at which no effective statistical interactions exist between particles with FES. In other words, the effective interactions between particles would be attractive for  $g < g_c$  and repulsive for  $g > g_c$ . Here  $g_c$  can

be self-consistently determined from the following equation

$$\rho = \int_0^\infty d\epsilon \mathcal{D}(\epsilon) \log \left[ 1 + \frac{1}{\omega(\epsilon, g_c)} \right], \quad (5.12)$$

where the chemical potential  $\mu$  can be obtained through  $\rho = \int_0^\infty d\epsilon \frac{\mathcal{D}(\epsilon)}{\omega(\epsilon, g_c, \mu) + g_c}$ , and the function  $\omega$  is determined by (2.9). We observe that  $g_c$  depends on the density  $\rho$ . The similar result has been reached in the case of constant DOS and for  $\alpha = 2$  a few years ago [16].

## 6 Thermodynamic properties

From (2.8), one may find that the average occupation number  $\langle n_{\mathbf{p}} \rangle$  is bounded by  $1/g$ , implying that  $\langle n_{\mathbf{p}} \rangle$  can not be *macroscopically* large for any momentum except  $g = 0$ , and thereby leading to the fact that no any condensation phenomenon occurs in the ideal gas obeying FES for  $0 < g \leq 1$ . For the free-particle energy given by (3.7), the Fermi energy for FES is

$$\epsilon_F = \left[ \frac{\rho g d}{\alpha A(\alpha, d)} \right]^{\frac{\alpha}{d}}. \quad (6.1)$$

To discuss the thermodynamic properties, we need to perform the Sommerfeld expansion. As a result, the expansion gives

$$\begin{aligned} I &= \int_0^\infty g(\epsilon) f(\epsilon) d\epsilon = \frac{G(\mu)}{\omega(0) + g} - \frac{G'(\mu)\mu}{\omega(0) + g} \\ &+ \frac{G''(\mu)\mu^2/2}{\omega(0) + g} + \frac{G'(\mu) - \mu G''(\mu)}{\beta} \log \left[ 1 + \frac{1}{\omega(0)} \right] \\ &+ \frac{G''(\mu)}{2\beta^2} [\omega(0) + g] \log^2 \left[ 1 + \frac{1}{\omega(0)} \right] \\ &+ \frac{G''(\mu)}{2\beta^2} \phi \left[ \log \left( 1 + \frac{1}{\omega(0)} \right) \right], \end{aligned} \quad (6.2)$$

where  $g(x)$  is any function of  $x$ , and

$$f(x) = \frac{1}{\omega(x) + g}, \quad (6.3)$$

$$G(x) = \int_0^x g(t) dt, \quad (6.4)$$

$$\phi(x) = \int_0^x \frac{t^2 e^t}{(e^t - 1)^2} dt, \quad (6.5)$$

with the notation  $G'(x) = dG(x)/dx$  and  $G''(x) = d^2G(x)/dx^2$ .  $\omega(0)$  is determined by the functional

$$\omega(0)^g [1 + \omega(0)]^{1-g} = e^{-\beta\mu}. \quad (6.6)$$

By using this expansion, the thermodynamic quantities can be obtained. For instance, the density  $\rho = N/V$  can be gained by setting  $g(\epsilon) = \mathcal{D}(\epsilon)$ . For fixed  $\rho$ ,  $\mu = \mu(g, T)$  can in turn be determined. By setting  $g(\epsilon) = \epsilon \mathcal{D}(\epsilon)$ , one

can get the internal energy  $E$  as

$$\begin{aligned} \frac{E}{V} &= \frac{A(\alpha, d)\mu^{\frac{d}{\alpha}+1}}{\omega(0) + g} \cdot \frac{d(d-2\alpha)}{2\alpha(\alpha+d)} \\ &+ \frac{A(\alpha, d)\mu^{\frac{d}{\alpha}}}{\beta} \left( 1 - \frac{d}{\alpha} \right) \log \left[ 1 + \frac{1}{\omega(0)} \right] \\ &+ \frac{A(\alpha, d)}{2\beta^2} \frac{d}{\alpha} \mu^{\frac{d}{\alpha}-1} \left\{ [\omega(0) + g] \log^2 \left[ 1 + \frac{1}{\omega(0)} \right] \right. \\ &\left. + \phi \left[ \log \left( 1 + \frac{1}{\omega(0)} \right) \right] \right\}. \end{aligned} \quad (6.7)$$

The specific heat per volume can thus be obtained *via*  $c_V(g, T) = (\frac{\partial E(T)/V}{\partial T})_{N, V}$ . The temperature-dependence of  $c_V(g, T)$  can be numerically determined by equations (6.6), (6.7) and  $\mu = \mu(g, T)$  self-consistently in general. At very low temperature,  $T \rightarrow 0$ ,  $\mu \approx \mu_0 \equiv \mu(T=0)$ ,  $\omega(0) \sim e^{-\beta\mu_0}$ , for  $0 < g \leq 1$  and  $g \neq 0$  one may find analytically

$$c_V(g, T) \sim \tilde{c}_0 T, \quad \tilde{c}_0 \approx \frac{\pi^2 d}{3\alpha} k_B^2 A(\alpha, d) \mu_0^{\frac{d}{\alpha}-1}, \quad (6.8)$$

where we have used  $\phi(\infty) = \pi^2/3$ . It suggests that at very low temperatures the specific heat depends linearly on temperature and goes to zero at  $T = 0$ , like an Fermi ideal gas.

For  $d = \alpha$ , *i.e.*, in the case of constant DOS, we can get a closed form for  $E/V$ :

$$\begin{aligned} \frac{E}{V} &= \frac{\rho^2}{2A(d, d)} \left( g + \frac{1}{e^{\frac{\rho}{A(d, d)k_B T}} - 1} \right) \\ &+ \frac{A(d, d)k_B^2 T^2}{2} \phi \left( \frac{\rho}{A(d, d)k_B T} \right). \end{aligned} \quad (6.9)$$

The specific heat  $c_V$  for fixed  $\rho$  is thus

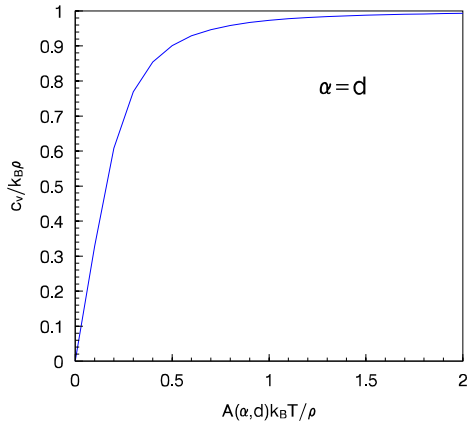
$$c_V(T) = A(d, d)k_B^2 T \phi \left( \frac{\rho}{A(d, d)k_B T} \right). \quad (6.10)$$

In this case, the specific heat is independent of  $g$  for fixed  $\rho$ , and goes to the classical value  $(k_B \rho)$  at  $T \rightarrow \infty$  as it should be. To see qualitatively the temperature-dependence of the specific heat, we depict equation (6.10) as an example, as shown in Figure 1.

The Sommerfeld expansion (6.2) could be used to investigate the magnetic susceptibility of an ideal gas obeying FES to see if the Pauli paramagnetism still sustains in the system apart from  $g = 1$ . This work is left to study in future.

## 7 Summary

In this paper we investigate systematically the quantum statistical mechanics of an ideal gas, with a general free-particle energy, obeying FES. Almost all basic properties of this simple system are studied. Some results appear for the first time, and some previously obtained results are



**Fig. 1.** The temperature-dependence of the specific heat for fixed density with constant DOS.

extended to cover more general cases. A general inequality among the pressure for arbitrary statistics parameter  $g$  and for the classical and Bose statistics is obtained for the fixed density, which could gain somewhat insight into the effective statistical interactions between particles with FES. For the density of states being constant, we present a theorem for the chemical potential and the pressure, which corrects some erroneous statements asserted by other people. A general relation between pressure and internal energy, say, the Bernoulli equation, is proved to hold for this system. The Virial expansion for this system is studied for a general case, and some wrong assertions by other people are also corrected. A critical value for  $g$ , below which the effective statistical interactions between particles are found to be attractive and above which repulsive, is predicted, and the equations used to determine  $g_c$  numerically are given. Finally, the Sommerfeld expansion is done, which allows to discuss the thermodynamic properties of the system. It is shown that at very low temperatures the specific heat depends linearly on temperature and vanishes at zero temperature. For the case with a constant density of states, a closed form for the specific heat is presented. We expect that the results obtained in this paper would be useful to further understand the characteristics of the particles (*e.g.*, anyons, semions, or quasi-particles in the fractional quantum Hall effect and in other low-dimensional electron or spin systems) with FES.

Moreover, there are a lot of interesting questions remaining open. For instance, how to construct a theory on QSM when weak interactions exist between particles obeying FES, or a generalized Landau Fermi liquid theory using FES, how to make the results obtained in this paper experimentally measurable, and if there also exists the de Haas-van Alphen effect in the system with FES, and so forth, deserve to discuss and to explore. Therefore, many fascinating but difficult problems remain in this new field.

#### Note added in proof

After acceptance of this manuscript, we were informed by S.B. Isakov, S. Ouvry and H. Ujino that the papers in

reference [20] are relevant to the present paper. We thank them for pointing this out to us

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17. We note that a similar theorem was stated in Ref. [15] for a general case. However, the proof presented by this author contains some serious errors, and thereby his statement is not justified, and could be incorrect.
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